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The behaviour of the O(3) sigma model at $\theta = \pi$

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Abstract. We investigate the claim that the O(3) sigma model at topological angle $\theta = \pi$ is conformally invariant. Using canonical methods we construct the current algebra and show that the Kac-Moody algebra does not emerge. This suggests that the model is not conformally invariant.

1. Introduction

Nonlinear sigma models in two dimensions have been the subject of investigation for many years. Classically they are conformally invariant, but very often a non-perturbative mass-gap is generated by quantum corrections. For the O(N) and CP(N) nonlinear sigma models this effect has been observed in the $1/N$ expansion [8], in lattice computations [9] and using exact S-matrix results [10]. The CP^N sigma model and the O(3) sigma model, which is equivalent to CP(1), can be extended to include a topological charge term. It has been speculated that the mass-gap of these extended models could vanish at topological angle $\theta = \pi$ [1, 2]. It is known that in the large N limit of the CP^N model the mass-gap does not vanish at any value of θ [3, 4]. Nevertheless, it has been argued that the mass-gap does vanish for the O(3) sigma model at $\theta = \pi$, the argument being based on a study of quantum spin chains [5]. Further, it is claimed that the fixed point for this model is the SU(2) Wess-Zumino-Witten (WZW) model at $k = 1$ [6]. The current algebra of the WZW model is known to be the Kac-Moody algebra. Using canonical quantisation we derive the current algebra for the O(3) sigma model with a topological charge term, and show that it does not reduce to a Kac-Moody algebra at $\theta = \pi$. Hence, it seems unlikely that this model is conformally invariant at $\theta = \pi$.

The plan of this paper is as follows. In section 2 we see how the current algebra can be calculated in the chiral SU(2) sigma model with a Wess-Zumino term and how in the WZW case the Kac-Moody algebra appears. In section 3, we calculate the current algebra in the O(3) sigma model at $\theta = \pi$ and see how it does not coincide with the diagonal algebra in the WZW model. Technical details for this calculation are given in the appendix. In section 4 we discuss operator ordering and its consequences for the current algebra, using Dirac quantisation. We see that in fact the algebra calculated classically is correct.

2. SU(2) chiral nonlinear sigma model with Wess-Zumino term

As we discussed in the introduction, the O(3) sigma model at $\theta = \pi$ is claimed to be equivalent to the SU(2) WZW model at $k = 1$. The WZW model provides an infrared

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stable fixed point for the SU(2) chiral sigma model with a Wess-Zumino term [7]. This result follows from the study of the current algebra of the later model [11], which at the fixed point becomes the Kac-Moody algebra.

The chiral SU(2) sigma model with a WZ term is defined by the action

$$S=\frac{1}{f^2}\int \mathrm{d}^2x\ \mathrm{Tr}[\partial^\mu U^\dagger\partial_\mu U] \\ +\frac{\mathrm{i}k}{24\pi}\int \mathrm{d}^3x\ \mathrm{Tr}[U^\dagger\partial_\mu UU^\dagger\partial_\nu UU^\dagger\partial_\lambda U]\varepsilon^{\mu\nu\lambda} \tag{2.1}$$

where U is an element of SU(2) and k must be an integer is the theory is to be well defined. The second term in the action is the so-called Wess-Zumino term. The action is invariant under $U'=G_LUG_R^{-1}$ where G_L and G_R are independent global SU(2) transformations.

In order to perform canonical quantisation we must go to a more conventional expression for the action, as a two-dimensional integral of a Lagrangian. On doing so we lose manifest invariance and the Lagrangian now changes by a total divergence under a symmetry transformation.

Now, with $U=\exp(\frac{1}{2}\mathrm{i}\lambda_a\phi^a)$ the Wess-Zumino term can be written as follows

$$\frac{1}{3}\int_D \mathrm{d}^2x\ \mathrm{Tr}[W_\mu W_\nu W_\lambda]\varepsilon^{\mu\nu\lambda}=\int_{\partial D}\mathrm{d}^2x\ h_{ab}\partial_\mu\phi^a\partial_\nu\phi^b\varepsilon^{\mu\nu} \tag{2.2}$$

where $W_\mu=U^\dagger\partial_\mu U$ and $h_{ab}=\mathrm{Tr}\int_0^1\mathrm{d}t\ \frac{1}{2}\lambda_e\phi^eW_{[a}W_{b]}$. This metric satisfies

$$3h_{[ab,c]}=-\tfrac{1}{2}\,\mathrm{Tr}[\lambda_{a'}\lambda_{b'}\lambda_{c'}R^{-1}_{a'[a}R^{-1}_{b'b}R^{-1}_{c'c]}] \tag{2.3}$$

where R^{-1} is the right vielbein ($U_{,a}=\frac{1}{2}\mathrm{i}\lambda_bUR^{-1}_{ba}$).

This last property is essential to the calculation. The other essential point is the following distribution identity:

$$a(y)\frac{\partial}{\partial y}\delta(x-y)=-a(x)\frac{\partial}{\partial x}\delta(x-y)+a'(y)\delta(x-y). \tag{2.4}$$

The symmetry currents for the right SU(2) are

$$j_{\mu a}=W_{\mu a}+\mathrm{i}\frac{f^2k}{16\pi}\varepsilon_{\mu\nu}W^\nu_a. \tag{2.5}$$

In the special case $f^2=16\pi/k$, the two components of the current are essentially the same and we have

$$j_+=\tfrac{1}{2}(j_{R0}+\mathrm{i}j_{R1})=j_{R0}. \tag{2.6}$$

The three commutation relations collapse to one, the Kac-Moody algebra:

$$[j_a,j_b]=\mathrm{i}f_{abc}j_c+\mathrm{i}\frac{k}{2\pi}\delta_{ab}\partial_x\delta(x-y). \tag{2.7}$$

Moreover, the current conservation $\partial^\mu j_{R\mu}$ becomes $\partial_-j_+=0$.

Similar results hold for the left SU(2) symmetry, which lead to another Kac-Moody algebra with opposite sign in the Schwinger term and $j_-=j_{L0}$ satisfying $\partial_+j=0$.

It is known that the unitary representation of a Kac-Moody algebra is conformally invariant. This implies that $\beta=0$ and therefore $f^2=16\pi/k$ is a fixed point of the renormalisation group.

If $k = 1$ the unitary representation is essentially unique [7]. Thus, a theory satisfying a Kac-Moody algebra with $k = 1$ must be equivalent to the wzw model at $k = 1$.

3. O(3) sigma model with a topological term

Let us consider the O(3) sigma model with a topological term included. This is given by the Lagrangian

$$L = \frac{1}{2g} \partial^{\mu} \phi \partial_{\mu} \phi + \frac{\theta}{8\pi} \phi (\partial_{\mu} \phi \times \partial_{\nu} \phi) \varepsilon^{\mu\nu} \tag{3.1}$$

with the additional constraint $\phi^2 = 1$.

This Lagrangian is invariant under SO(3) transformations. In two dimensions

$$Q = \frac{1}{8\pi} \phi (\partial_{\mu} \phi \times \partial_{\nu} \phi) \varepsilon^{\mu\nu} \tag{3.2}$$

is the winding number of the map $S^2 \mapsto S^2$ defined by $\phi(x)$. It is an integer and it is constant under small variations of the fields.

Although the Lagrangian is not parity invariant for a general θ, it is invariant in the two special cases $\theta = 0$ and $\theta = \pi$. This is a consequence of the fact that physics, depending only on $\exp(\mathrm{i}S)$, is periodic in θ. Therefore a parity transformation, which effectively changes the sign of θ, does not change the weight of a field configuration in the action functional if $\theta = \pi$. Moreover, the symmetry becomes the full O(3) due to the extra symmetry $\phi \mapsto -\phi$. It is worth noticing that Q, being a constant under small variations, does not affect perturbation theory and, as a consequence, θ does not get renormalised perturbatively.

Now we are interested in seeing what the consequences of conformal invariance in such system would be. We see that this implies equivalence with the wzw model at $k = 1$, as this is the only conformally invariant system with a continuous symmetry commuting with parity and some other discrete Z_2 symmetry (which in this case is $\phi \mapsto -\phi$).

The canonical computation of the current algebra in the chiral SU(2) sigma model with a Wess-Zumino term shows us that the Kac-Moody algebra appears for the special value of the coupling constant that defines the wzw model. This constrains the system to be conformally invariant and the states to fit some unitary representation. There is essentially only one unitary representation for $k = 1$.

In our model we again have two coupling constants, one of them not renormalised. We would expect a similar effect in the current algebra, leading to a Kac-Moody algebra, if the system is equivalent to the wzw model.

In order to perform the computation of the algebra using canonical methods we first solve the constraint. To do so we change parametrisation and introduce the fields φ_a, $a = 1, 2$, as follows:

$$\phi_a = \frac{\varphi_a}{(1 + \frac{1}{4}\varphi^2)^2} \qquad \phi_3 = \frac{1 - \frac{1}{4}\varphi^2}{1 + \frac{1}{4}\varphi^2}. \tag{3.3}$$

The Lagrangian becomes:

$$L = \frac{1}{2} \frac{1}{(1 + \frac{1}{4}\varphi^2)^2} \left(\frac{1}{g^2} \partial^{\mu} \varphi_a \partial_{\mu} \varphi_a + \frac{\theta}{4\pi} \varepsilon^{\mu\nu} \partial_{\mu} \varphi_a \partial_{\nu} \varphi_a \varepsilon_{ab} \right). \tag{3.4}$$

The drawback of this choice is that the symmetry does not act linearly in the fields φ. It still acts linearly in the ϕ_3 direction:

$$\delta_3\varphi_a = \varepsilon_{ab}\varphi_b \tag{3.5}$$

but it becomes nonlinear in the other two directions:

$$\delta_a\varphi_b = (1-\tfrac{1}{4}\varphi^2)\delta_{ab} + \tfrac{1}{2}\varphi_a\varphi_b. \tag{3.6}$$

The number of commutators that we have to compute is nine in the unconstrained formalism. But when expressed in terms of the original fields ϕ they reduce to three independent ones.

The current algebra for the $\theta = 0$ case is (expressed in terms of the ϕ's)

$$\begin{aligned}
&[j_0^i, j_0^j] = -\mathrm{i}\varepsilon^{ijk} j_0^k \\
&[j_0^i, j_1^j] = -\mathrm{i}\varepsilon^{ijk} j_1^k - \mathrm{i}\frac{1}{g}(\delta^{ij} - \phi^i\phi^j)\frac{\partial}{\partial x}\delta(x-y) \\
&[j_1^i, j_1^j] = 0.
\end{aligned} \tag{3.7}$$

We note that the commutator of j_0 with j_1 contains a field dependent term, which is in fact an O(3) projector.

The introduction of the topological term does not change the equations of motion, since it is a constant under small perturbations, but it changes the momenta and the currents, adding to them a term which is trivially conserved:

$$j'^a_\mu = \varepsilon_{\mu\nu}\partial^\nu\Psi^a \tag{3.8}$$

where Ψ^a are arbitrary functions.

In order to simplify the calculation we re-express the Lagrangian as

$$L = \frac{1}{2g^2}\frac{1}{(1+\frac{1}{4}\varphi^2)^2}(\partial^\mu\varphi_a\partial_\mu\varphi_a + a\varepsilon^{\mu\nu}\partial_\mu\varphi_a\partial_\nu\varphi_a\varepsilon_{ab}) \tag{3.9}$$

and we consider a general real a. The momenta are

$$\Pi_a = \frac{1}{g^2}\frac{1}{(1+\frac{1}{4}\varphi^2)^2}[\dot\varphi_a + a\varepsilon_{ab}\varphi'_b] \tag{3.10}$$

and the currents are

$$\begin{aligned}
&j_0^3 = -\varepsilon_{ab}\varphi^a\Pi^b \\
&j_0^a = (1+\tfrac{1}{4}\varphi^2)\Pi^a - \tfrac{1}{2}\varepsilon_{ab}\varphi^a j_0^3 \\
&j_1^3 = -a\varphi^c\Pi^c + (a^2+1)j^3_{1\theta=0} \\
&j_1^a = -a\varepsilon_{ab}[(1+\tfrac{1}{4}\varphi^2)\Pi^b - \tfrac{1}{2}(\Pi^c\varphi^c)\varphi^b] + (a^2+1)j^a_{1\theta=0}.
\end{aligned} \tag{3.11}$$

The zero component of the currents have the same expression as they had when $\theta = 0$ if written in terms of the momentum. This ensures the right algebra for the zero components and therefore for the charges. There is momentum dependence in the j_i^1, as we would expect if we want a Kac–Moody algebra to be satisfied. But we see that with such momentum dependence we will not get rid of the field dependent term of the algebra. In fact, the following algebra emerges:

$$
\begin{aligned}
&[j_0^i, j_0^j] = -\mathrm{i}\varepsilon^{ijk} j_0^k \\
&[j_0^i, j_1^j] = -\mathrm{i}\varepsilon^{ijk} j_1^k - \mathrm{i}\frac{1}{g^2}(1+a^2)(\delta^{ij} - \phi^i\phi^j)\frac{\partial}{\partial x}\delta(x-y) \\
&[j_1^i, j_1^j] = \mathrm{i}a^2\varepsilon^{ijk} j_0^k - \mathrm{i}\frac{a}{g^2}(1+a^2)\varepsilon^{ijk}\phi'^k.
\end{aligned}
\tag{3.12}
$$

4. Constrained formalism

The calculation in the previous section has been classical. We have calculated the current algebra using Poisson brackets and we have not worried about operator ordering. These are not trivial problems in the unconstrained formalism, as we have operators in the denominator, which can only be dealt with in a power series expansion.

The problem is clarified when we use the Dirac procedure of quantising systems with constraints. The Dirac brackets arising from the Lagrangian with a topological term are the same as the ones arising for the system at $\theta=0$. Ordering problems in both current algebra and the Dirac brackets can be solved by demanding hermiticity.

Classically, the constraints that are relevant to our calculation are

$$
\phi^2 = 0 \qquad \phi\Pi = 0. \tag{4.1}
$$

The Dirac brackets are

$$
\begin{aligned}
&[\phi^i(x), \phi^j(y)] = 0 \\
&[\phi^i(x), \Pi^j(y)] = \mathrm{i}(\delta^{ij} - \phi^i\phi^j)\delta(x-y) \\
&[\Pi^i(x), \Pi^j(x)] = \mathrm{i}(\Pi^i\phi^j - \Pi^j\phi^i)\delta(x-y)
\end{aligned}
\tag{4.2}
$$

and the currents are (we use vector notation here)

$$
j_\mu = \phi\times\partial_\mu\phi - a\varepsilon_{\mu\nu}[\phi\times(\phi\times\partial^\nu\phi)]. \tag{4.3}
$$

We now give a precise ordering by demanding hermiticity. For the constraints this will mean

$$
\phi^2 = 0 \qquad \tfrac{1}{2}(\phi\Pi + \Pi\phi) = 0. \tag{4.4}
$$

The commutation relations are consistent as written before.

In order to be able to construct Hermitian currents, we have to re-express them in terms of the momenta, as we only know the commutation relations among the ϕ's and the Π's, and we do not know how the $\dot{\phi}$ behave. We have

$$
j_0 = -\phi\times\Pi \tag{4.5}
$$

which is well defined and Hermitian, and

$$
j_1 = \frac{1}{g^2}[\phi\times\phi' + a(\phi\times(\phi\times(\Pi - a\phi\times\phi')))]. \tag{4.6}
$$

The only term which is not well defined is $\phi\times(\phi\times\Pi)$, as ϕ and ϕ' commute. The only Hermitian combination for this term is

$$\phi\times(\phi\times\Pi)^i=\tfrac{1}{2}\varepsilon^{ijk}\varepsilon^{klm}(\Pi^m\phi^j\phi^l+\phi^j\phi^l\Pi^m)=-\Pi^i \tag{4.7}$$

where we have made explicit use of both constraints (4.4). We are left with

$$j_1=a\Pi-\frac{1}{g^2}(1+a^2)\phi\times\phi'. \tag{4.8}$$

Now we will be able to calculate the current algebra using the commutation relations arising from the Dirac brackets and treating the fields as non-commuting operators. The algebra is unchanged, and when we rewrite it in terms of the original coupling constants we have

$$\begin{aligned}
&[j_0^i,j_0^j]=-\mathrm{i}\varepsilon^{ijk}j_0^k\\
&[j_0^i,j_1^j]=-\mathrm{i}\varepsilon^{ijk}j_1^k-\mathrm{i}\frac{1}{g^2}\left(1+\frac{\theta^2g^4}{16\pi^2}\right)(\delta^{ij}-\phi^i\phi^j)\frac{\partial}{\partial x}\delta(x-y)\\
&[j_1^i,j_1^j]=\mathrm{i}\frac{\theta^2g^4}{16\pi^2}\varepsilon^{ijk}j_0^k-\mathrm{i}\frac{\theta}{4\pi}\left(1+\frac{\theta^2g^4}{16\pi^2}\right)\varepsilon^{ijk}\phi'^k.
\end{aligned} \tag{4.9}$$

As discussed in section 3, that algebra does not reduce to the Kac–Moody algebra for any value of the topological angle θ. Thus we should study its representations at different values of the topological angle. In fact, we know that for $\theta=2n\pi$, the representations should be the same as the ones at $\theta=0$. This leads us to believe that representations at different values of θ are going to be essentially the same.

In conclusion, we have shown how the Kac-Moody emerges in the wz SU(2) chiral model for a special value of the coupling constant. We expected a similar effect in the O(3) sigma model at $\theta=\pi$ if the two models are equivalent. We have calculated the algebra in the O(3) sigma model at $\theta=\pi$ and we have seen there is not any value of the coupling constants for which the Kac–Moody algebra appear.

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Appendix. Computation of the current algebra for the O(3) sigma model with a topological term

As the currents for the model with a topological term can be expressed in a simple way in terms of the currents of the model without the topological term (corresponding to $a=0$ in (3.9)), it is convenient to calculate the algebra for the former model first. The complete calculation becomes then quite simple.

In both cases the Lagrangian is invariant under

$$\delta\varphi_a=\alpha\delta_3\varphi_a \qquad \delta\varphi_a=\alpha_b\delta_b\varphi_a \tag{A1}$$

for constants α and α_a, where $\delta_3\varphi_a$ and $\delta_b\varphi_a$ are defined as in (3.5) and (3.6). The choice of $\delta_b\varphi_a$ is not the one that naturally arises from $\delta_i\phi_j = \varepsilon_{ijk}\phi_k$, but it is simply related with it:

$$\delta_b\varphi_a = -\varepsilon_{ab}\delta'_b\varphi_a. \tag{A2}$$

Commutation relations are the same for δ and δ', and the calculation is simpler using δ.

It is convenient to define

$$\Pi_a^\mu = \frac{\partial L}{\partial(\partial_\mu\varphi_a)} \qquad \Pi_a^0 = \Pi_a. \tag{A3}$$

In order to find the currents we consider local variations of the Lagrangian. We get

$$\delta L = \Pi_a^\mu \partial_\mu \alpha \delta\varphi_a \tag{A4}$$

and therefore the currents are

$$j_3^\mu = \Pi_a^\mu \delta_3\varphi_a \qquad j_a^\mu = \Pi_b^\mu \delta_a\varphi_b. \tag{A5}$$

Poisson brackets are

$$[\varphi_a(x), \Pi_b(y)] = \mathrm{i}\delta_{ab}\delta(x-y). \tag{A6}$$

The zero components of the currents have the same form with or without the topological term:

$$j_3^0 = -\varepsilon_{ab}\varphi_a\Pi_b \tag{A7}$$

$$j_a^0 = (1+\tfrac{1}{4}\varphi^2)\Pi_a - \tfrac{1}{2}\varepsilon_{ab}\varphi_a j_3^0. \tag{A8}$$

The algebra for these terms is easy to compute. We get

$$[j_3^0, j_a^0] = -i\varepsilon_{ab} j_b^0 \qquad [j_a^0, j_b^0] = -\mathrm{i}\varepsilon_{ab} j_3^0 \tag{A9}$$

which can be written

$$[j_i^0, j_j^0] = -\mathrm{i}\varepsilon_{ijk} j_k^0. \tag{A10}$$

The calculation of the j^0, j^1 commutators is more involved due to the presence of spatial derivatives of the fields. We consider first the $a=0$ case. Then we have

$$\Pi_a^1 = -\frac{1}{g^2}\frac{\varphi'}{(1+\frac{1}{4}\varphi^2)^2}.$$

We need to use equation (2.4) and we get

$$\begin{aligned}
[j_3^0, j_3^1] &= -\frac{\mathrm{i}}{g}\frac{\varphi^2}{(1+\frac{1}{4}\varphi^2)^2}(y)\frac{\partial}{\partial x}\delta(x-y) \\
[j_3^0, j_a^1] &= -\mathrm{i}\varepsilon_{ab} j_b^1 - \frac{\mathrm{i}}{g}\frac{\varepsilon_{ab}\varphi_b(1+\frac{1}{4}\varphi^2)}{(1+\frac{1}{4}\varphi^2)^2}(y)\frac{\partial}{\partial x}\delta(x-y) \\
[j_a^0, j_3^1] &= \mathrm{i}\varepsilon_{ab} j_b^1 - \frac{\mathrm{i}}{g^2}\frac{\varepsilon_{ab}\varphi_b(1+\frac{1}{4}\varphi^2)}{(1+\frac{1}{4}\varphi^2)^2}(y)\frac{\partial}{\partial x}\delta(x-y) \\
[j_a^0, j_b^1] &= -\mathrm{i}\varepsilon_{ab} j_b^1 + \frac{\mathrm{i}}{g^2}\left(\delta_{ab} - \varepsilon_{ac}\varepsilon_{bd}\frac{\varphi_c\varphi_d}{(1+\frac{1}{4}\varphi^2)^2}\right)\frac{\partial}{\partial x}\delta(x-y)
\end{aligned} \tag{A11}$$

which can be rewritten, in terms of the constraint fields:

$$[j_i^0, j_j^1] = -\mathrm{i}\varepsilon_{ijk} j_k^1 - \frac{\mathrm{i}}{g^2}(\delta_{ij} - \phi_i\phi_j)\frac{\partial}{\partial x}\delta(x-y). \tag{A12}$$

To re-express the commutator like this we must make use of δ' instead of δ (see (A2)). The spatial components of the currents commute, as they do not depend on the momenta.

Now we consider a non-zero a. In that case we have a different Π_a^1:

$$\Pi_a^1 = \frac{1}{g}\frac{1}{(1+\frac{1}{4}\varphi^2)^2}[\varphi'_a + a\varepsilon_{ab}\dot{\varphi}_b] = (1+a^2)\Pi^1_{a\theta=0} - a\varepsilon_{ab}\Pi_b \tag{A13}$$

then

$$j_i^1 = \Pi_a^1 \delta_i \varphi_a = -a^2\varepsilon_{ab}\Pi_b\delta_i\varphi_a + (1+a^2) j^1_{i\theta=0} \tag{A14}$$

and therefore

$$j_3^1 = -a\varphi_c\Pi_c + (a^2+1) j^1_{3\theta=0} \tag{A15}$$

$$j_a^1 = -a\varepsilon_{ab}[(1+\tfrac{1}{4}\varphi^2)\Pi_b - \tfrac{1}{2}(\Pi_c\varphi_c)\varphi_b] + (a^2+1) j^1_{a\theta=0}. \tag{A16}$$

We see that the dependence in φ' is included in the $j^1_{i\theta=0}$ part of the currents. This means that the calculation of $[j_i^0, j_j^1]$ is now trivial.

We are left with the computation of $[j_i^1, j_j^1]$. In two dimensions there can be no Schwinger term in this commutator [12]. This result is confirmed by explicit calculation. We get

$$\begin{aligned} [j_3^1, j_a^1] &= \mathrm{i}a^2\varepsilon_{ab} j_b^0 - \frac{\mathrm{i}a}{g^2}(1+a^2)\varepsilon_{ab}\phi'_b \\ [j_a^1, j_b^1] &= \mathrm{i}a^2\varepsilon_{ab} j_3^0 - \frac{\mathrm{i}a}{g^2}(1+a^2)\varepsilon_{ab}\phi'_3 \end{aligned} \tag{A17}$$

which again can be expressed

$$[j_i^1, j_j^1] = \mathrm{i}a^2\varepsilon_{ijk} j_k^0 - \frac{\mathrm{i}a}{g^2}(1+a^2)\varepsilon_{ijk}\phi'_j.$$

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